SECTION 3.5: THE DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

DERIVATIVES OF THE CIRCULAR FUNCTIONS:

- $D_x[\sin(x)] = \cos(x)$
- $D_x[\sec(x)] = \sec(x)\tan(x)$
- $D_x[\tan(x)] = \sec^2(x)$
- $D_x[\cos(x)] = -\sin(x)$
- $D_x[\csc(x)] = -\csc(x)\cot(x)$
- $D_x[\cot(x)] = -\csc^2(x)$

We'll show the proof of three of these formulas; the rest are left to you as exercises.

PROOF: $D_x[\sin(x)] = \cos(x)$

$$\begin{aligned}
D_x[\sin(x)] &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) - \sin(x) + \cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \left[\sin(x)\,\frac{\cos(h)-1}{h} + \cos(x)\,\frac{\sin(h)}{h}\right] \\
&= \lim_{h\to 0} \left[\sin(x)\,\frac{\cos(h)-1}{h}\right] + \lim_{h\to 0} \left[\cos(x)\,\frac{\sin(h)}{h}\right] \\
&= \sin(x)\,\lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x)\,\lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \sin(x)\cdot 0 + \cos(x)\cdot 1 && \text{Since } \lim_{h\to 0} \frac{\cos(h)-1}{h} = 0 \text{ and } \lim_{h\to 0} \frac{\sin(h)}{h} = 1 \\
D_x[\sin(x)] &= \cos(x)
\end{aligned}$$

A similar argument shows $D_x[\cos(x)] = -\sin(x)$.

A you may expect, once we know the derivatives of $\sin(x)$ and $\cos(x)$, we are able to obtain the derivative formulas for the remaining circular functions using the Reciprocal and Quotient identities learned in Trigonometry along with the Quotient Rule for derivatives.

PROOF: $D_x[\csc(x)] = -\csc(x)\cot(x)$:

$$\begin{aligned}
D_x[\csc(x)] &= D_x\left[\frac{1}{\sin(x)}\right] \\
&= \frac{\sin(x)D_x[1]-(1)D_x[\sin(x)]}{(\sin(x))^2} && \text{Quotient Rule} \\
&= \frac{\sin(x)\cdot 0 - 1\cdot\cos(x)}{(\sin(x))^2} && D_x[1]=0 \text{ and } D_x[\sin(x)]=\cos(x) \\
&= \frac{-\cos(x)}{(\sin(x))^2} \\
&= -\frac{1}{\sin(x)}\frac{\cos(x)}{\sin(x)} \\
D_x[\csc(x)] &= -\csc(x)\cot(x)
\end{aligned}$$

A similar argument shows $D_x[\sec(x)] = \sec(x)\tan(x)$.

PROOF: $D_x[\tan(x)] = \sec^2(x)$:

$$\begin{aligned}
D_x[\tan(x)] &= D_x\left[\frac{\sin(x)}{\cos(x)}\right] \\
&= \frac{\cos(x)D_x[\sin(x)]-\sin(x)D_x[\cos(x)]}{(\cos(x))^2} && \text{Quotient Rule} \\
&= \frac{\cos(x)\cdot\cos(x)-\sin(x)\cdot(-\sin(x))}{(\cos(x))^2} && D_x[\sin(x)]=\cos(x) \text{ and } D_x[\cos(x)]=-\sin(x) \\
&= \frac{\cos^2(x)+\sin^2(x)}{(\cos(x))^2} \\
&= \frac{1}{\cos^2(x)} && \cos^2(x)+\sin^2(x)=1 \\
D_x[\tan(x)] &= \sec^2(x)
\end{aligned}$$

A similar argument shows $D_x[\cot(x)] = -\csc^2(x)$.

EXAMPLE 1: (VIDEO) Find and simplify the indicated derivative:

1. For $f(x) = 5\cos(x)$, find and simplify $f'(x)$.

Ans: $f'(x) = -5\sin(x)$

2. For $y = x^2\sin(x)$, find and simplify $\dfrac{dy}{dx}$.

Ans: $\dfrac{dy}{dx} = 2x\sin(x) + x^2\cos(x)$

3. For $F(\theta) = \dfrac{\sin(\theta)}{1+\cos(\theta)}$, find and simplify $F'(\theta)$.

Ans: $F'(\theta) = \dfrac{1}{1+\cos(\theta)}$

EXAMPLE 2: (VIDEO) Find and simplify the indicated derivative:

1. Find and simplify: $D_x[\sqrt{x}\sec(x)]$.

Ans: $D_x[\sqrt{x}\sec(x)] = \dfrac{\sec(x)}{2x^{1/2}} + x^{1/2}\sec(x)\tan(x)$

2. For $y = \dfrac{\tan(t)}{t^2+1}$, find and simplify $\dfrac{dy}{dt}$.

Ans: $\dfrac{dy}{dt} = \dfrac{t^2\sec^2(t) - 2t\tan(t) + \sec^2(t)}{(t^2+1)^2}$

3. Find $D_t\left[\dfrac{\cot(t)}{\csc(t)}\right]$.

Ans: $D_t\left[\dfrac{\cot(t)}{\csc(t)}\right] = -\sin(t)$, provided $\sin(t) \neq 0$.

EXAMPLE 3: (VIDEO) Find the equation of the tangent line to $y = \sin(x) - x\cos(x)$ at $x = \dfrac{\pi}{2}$. Check your answer using a graphing utility.

Ans: $y = \dfrac{\pi}{2}x - \dfrac{\pi^2}{4} + 1$

EXAMPLE 4: (VIDEO) Find $D_x\left[\dfrac{x\cos(x)}{x^2+1}\right]$.

Ans: $D_x\left[\dfrac{x\cos(x)}{x^2+1}\right] = \dfrac{-x^3\sin(x) - x^2\cos(x) - x\sin(x) + \cos(x)}{(x^2+1)^2}$

EXPLORATION: Let $f(x) = \sin(x)$.

- Find $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, $f^{(5)}(x)$, $f^{(6)}(x)$, $f^{(7)}(x)$, $f^{(8)}(x)$. What pattern emerges?

$f'(x) =$

$f''(x) =$

$f'''(x) =$

$f^{(4)}(x) =$

$f^{(5)}(x) =$

$f^{(6)}(x) =$

$f^{(7)}(x) =$

$f^{(8)}(x) =$

- Find $D_x^{117}[\sin(x)]$.

HOMEWORK: Section 3.5: 23 - 51 odd, 55 - 63 odd, 73, 75